

FINITE-GAP MINIMAL LAGRANGIAN SURFACES IN \mathbb{CP}^2

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ABSTRACT. In this paper we suggest a method for constructing minimal Lagrangian immersions of \mathbb{R}^2 in \mathbb{CP}^2 with induced diagonal metric in terms of Baker–Akhiezer functions of algebraic curves.

INTRODUCTION

We propose new approach for constructing minimal Lagrangian (**ML**) surfaces in \mathbb{CP}^2 in terms of Baker–Akhiezer functions of algebraic curves. This approach is based on the work [1].

It is well known that if we choose the conformal coordinates on a **ML**-torus in \mathbb{CP}^2 with the induced metric $ds^2 = e^{v(x,y)}(dx^2 + dy^2)$, then the function $v(x,y)$ satisfies the Tzitzeica equation (see [2]). This equation allows the Lax representation with a spectral parameter, which was found by Mikhailov [3]. Sharipov [4], using the Lax representation, constructed the finite-gap solutions of the Tzitzeica equation, and the solutions expressed in terms of the theta-functions of the trigonal spectral curves, which allow the holomorphic involution. The existence of periodic solutions among quasiperiodic solutions is shown in [5].

In fact, the conformal coordinates are not always suitable for the description of **ML**-tori in \mathbb{CP}^2 . To confirm this, let us consider the following example. Let K denote a cone in \mathbb{R}^3 , defined by the equation

$$mu_1^2 + nu_2^2 = (m+n)u_3^2,$$

where $m, n \in \mathbb{Z}, m, n > 0$. Let \tilde{K} denote the intersection of K with the unit sphere

$$u_1^2 + u_2^2 + u_3^2 = 1.$$

We construct the mapping ψ from $\tilde{K} \times S^1$ in \mathbb{CP}^2 as a composition $\psi = \mathcal{H} \circ \tilde{\varphi}$, where

$$\tilde{\varphi} : \tilde{K} \times S^1 \rightarrow S^5, \quad \tilde{\varphi}(P) = (u_1 e^{\pi i m y}, u_2 e^{\pi i n y}, u_3 e^{\pi i (m+n)y}),$$

\mathcal{H} — Hopf projection $\mathcal{H} : S^5 \rightarrow \mathbb{CP}^2$, $P \in \tilde{K} \times S^1$, y is a coordinate on S^1 .

The image of ψ is **ML**-torus if the involution

$$(v_1, v_2) \rightarrow (v_1 \cos(n\pi), v_2 \cos(m\pi))$$

preserve the orientation of the ellipse

$$mv_1^2 + nv_2^2 = m + n$$

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and **ML**-Klein bottle if it doesn't (see [6],[7]). This surface can be defined as an image of the composition $\mathcal{H} \circ \varphi$, where $\varphi : \mathbb{R}^2 \rightarrow S^5$,

$$\varphi(x, y) = \left(\frac{\sin(x)\sqrt{m+n}}{\sqrt{2m+n}} e^{\pi i m y}, \right. \\ \left. \frac{\cos(x)\sqrt{m+n}}{\sqrt{m+2n}} e^{\pi i n y}, \sqrt{\frac{n \cos^2(x)}{m+2n} + \frac{m \sin^2(x)}{2m+n}} e^{-\pi i (m+n)y} \right)$$

(probably these tori coincide with the tori from [8], where these tori are described in conformal coordinates). In coordinates x, y the induced metric has a diagonal form

$$ds^2 = 2e^{v_1} dx^2 + 2e^{v_2} dy^2,$$

and one can ascertain that $v_1 \neq v_2$. Thus, on the one hand **ML**-tori correspond to the periodic solutions of the Tzitzeica equation (all of these solutions can be expressed in terms of the theta-function of the spectral curves), and on the other hand this example demonstrates that there exist coordinates x, y , in which the metric is diagonal and which are more suitable for the description of this **ML**-torus, as in these coordinates the tori are described in elementary functions.

In this paper we construct **ML**-mapping of the plane in \mathbb{CP}^2 (with induced diagonal metric) by the spectral data, which are easier than spectral data for the solution of the Tzitzeica equation. We construct such mapping by the real algebraic curve, which allows a holomorphic involution. Specifically, we do not require the spectral curve to be trigonal (as for the solutions of the Tzitzeica equation).

The main difference of our method from the method of [4] consists in the following. We do not use the Lax representation with a spectral parameter of the Tzitzeica equation. Instead of this we construct the explicit mapping

$$\varphi : \mathbb{R}^2 \rightarrow S^5 \subset \mathbb{C}^3,$$

which satisfies the equations

$$\langle \varphi, \varphi_x \rangle = \langle \varphi, \varphi_y \rangle = \langle \varphi_x, \varphi_y \rangle = 0, \quad (1)$$

where $\langle \cdot, \cdot \rangle$ — Hermitian product in \mathbb{C}^3 . A composition of the mappings $\varphi \circ \mathcal{H}$ gives a Lagrangian mapping of the plane in \mathbb{CP}^2 . By means of the corollary 1 we obtain the minimal mappings.

Note that in this paper we do not discuss the problem of existence of the smooth periodic solutions. As the spectral curve is hyperelliptic, the methods of the paper [9] can be used to prove the existence of the periodic solutions.

In section 1 we get the equations of **ML**-mapping of a plane in \mathbb{CP}^2 with diagonal metric. In section 2 we remind the definition of the Baker–Akhiezer function. In section 3 we prove the main theorem (Theorem 2) and give the example of **ML**-sphere, corresponding to the reducible rational spectral curve.

1. EQUATIONS OF LAGRANGIAN SURFACES WITH THE DIAGONAL METRIC

Let us introduce the following notations

$$|\varphi_x|^2 = 2e^{v_1(x,y)}, \quad |\varphi_y|^2 = 2e^{v_2(x,y)}.$$

Then from (1) it follows, that the matrix

$$\tilde{\Phi} = \left(\varphi, \frac{1}{\sqrt{2}} e^{-\frac{v_1}{2}} \varphi_x, \frac{1}{\sqrt{2}} e^{-\frac{v_2}{2}} \varphi_y \right)^\top$$

belongs to the group $\mathbf{U}(\mathbf{3})$. A *Lagrangian angle* $\beta(x, y)$ is a function defined from the equality

$$e^{i\beta(x, y)} = \det \tilde{\Phi}.$$

The Lagrangian angle defines the mean curvature vector H of the Lagrangian surface

$$H = J\nabla\beta,$$

where J is the complex structure on $\mathbb{C}P^2$. Consequently, if $\beta = \text{const}$, then the surface is minimal.

From the definition of the Lagrangian angle we get

$$\Phi = \begin{pmatrix} \varphi^1 & \varphi^2 & \varphi^3 \\ \frac{1}{\sqrt{2}} e^{-\frac{v_1}{2} - i\frac{\beta}{2}} \varphi_x^1 & \frac{1}{\sqrt{2}} e^{-\frac{v_1}{2} - i\frac{\beta}{2}} \varphi_x^2 & \frac{1}{\sqrt{2}} e^{-\frac{v_1}{2} - i\frac{\beta}{2}} \varphi_x^3 \\ \frac{1}{\sqrt{2}} e^{-\frac{v_2}{2} - i\frac{\beta}{2}} \varphi_y^1 & \frac{1}{\sqrt{2}} e^{-\frac{v_2}{2} - i\frac{\beta}{2}} \varphi_y^2 & \frac{1}{\sqrt{2}} e^{-\frac{v_2}{2} - i\frac{\beta}{2}} \varphi_y^3 \end{pmatrix} \in \mathbf{SU}(\mathbf{3}).$$

The matrix Φ satisfies the equations

$$\Phi_x = A\Phi, \quad \Phi_y = B\Phi, \quad (2)$$

where matrices $A, B \in \mathbf{su}(\mathbf{3})$ have the form

$$A = \begin{pmatrix} 0 & \sqrt{2} e^{\frac{v_1}{2} + i\frac{\beta}{2}} & 0 \\ -\sqrt{2} e^{\frac{v_1}{2} - i\frac{\beta}{2}} & if & \frac{1}{2} e^{\frac{v_1}{2} - \frac{v_2}{2}} (2ih - v_{1_y} + i\beta_y) \\ 0 & \frac{1}{2} e^{\frac{v_1}{2} - \frac{v_2}{2}} (2ih + v_{1_y} + i\beta_y) & -if \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & \sqrt{2} e^{\frac{v_2}{2} + i\frac{\beta}{2}} \\ 0 & ih & \frac{1}{2} e^{\frac{v_2}{2} - \frac{v_1}{2}} (i\beta_x - 2if + v_{2_x}) \\ -\sqrt{2} e^{\frac{v_2}{2} - i\frac{\beta}{2}} & \frac{1}{2} e^{\frac{v_2}{2} - \frac{v_1}{2}} (i\beta_x - 2if - v_{2_x}) & -ih \end{pmatrix},$$

$f(x, y)$ and $h(x, y)$ being some real functions. The equations (2) implies the following equations

$$\varphi_{xx} = \Gamma_{11}^1 \varphi_x + \Gamma_{11}^2 \varphi_y + b_{11} \varphi,$$

$$\varphi_{xy} = \Gamma_{12}^1 \varphi_x + \Gamma_{12}^2 \varphi_y + b_{12} \varphi,$$

$$\varphi_{yy} = \Gamma_{22}^1 \varphi_x + \Gamma_{22}^2 \varphi_y + b_{22} \varphi,$$

where

$$\Gamma_{11}^1 = \frac{1}{2} (2if + v_{1_x} + i\beta_x), \quad \Gamma_{11}^2 = \frac{1}{2} e^{v_1 - v_2} (2ih - v_{1_y} + i\beta_y),$$

$$\Gamma_{12}^1 = \frac{1}{2} (2ih + v_{1_y} + i\beta_y), \quad \Gamma_{12}^2 = \frac{1}{2} (-2if + v_{2_x} + i\beta_x),$$

$$\Gamma_{22}^1 = \frac{1}{2} e^{v_2 - v_1} (-2if - v_{2_x} + i\beta_x), \quad \Gamma_{22}^2 = \frac{1}{2} (-2ih + v_{2_y} + i\beta_y),$$

$$b_{11} = -2e^{v_1}, \quad b_{12} = 0, \quad b_{22} = -2e^{v_2}.$$

From this it follows the key lemma of our construction.

Lemma 1. *The following equalities hold:*

$$\begin{aligned}\Gamma_{11}^1 + \Gamma_{12}^2 &= \frac{1}{2}(v_{1x} + v_{2x}) + i\beta_x, \\ \Gamma_{12}^1 + \Gamma_{22}^2 &= \frac{1}{2}(v_{1y} + v_{2y}) + i\beta_y.\end{aligned}$$

From lemma 1 we get

Corollary 1. *If*

$$\operatorname{Im}(\Gamma_{11}^1 + \Gamma_{12}^2) = \operatorname{Im}(\Gamma_{12}^1 + \Gamma_{22}^2) = 0,$$

then the surface is minimal.

The corollary 1 gives us the condition for surface to be minimal in a diagonal metric.

2. BAKER–AKHIEZER FUNCTION

In this paragraph we remind the definition of two-point Baker–Akhiezer function. By means of this function we construct **ML**-mapping of the plane in \mathbb{CP}^2 .

Let Γ be a Riemann surface of genus g (actually the following construction can be generalized on singular algebraic curves over \mathbb{C}). Suppose that the divisor

$$\gamma = \gamma_1 + \dots + \gamma_g,$$

is given on Γ , r , $P_1, P_2 \in \Gamma$ are fixed points, and k_1^{-1}, k_2^{-1} are local parameters in the neighborhoods of the points P_1 and P_2 . The *two-point Baker–Akhiezer function*, corresponding to the *spectral data*

$$\{\Gamma, P_1, P_2, k_1, k_2, \gamma, r\},$$

is a function $\psi(x, y, P)$, $P \in \Gamma$, with the following characteristics:

1) in the neighborhoods of P_1 and P_2 the function ψ has essential singularities of the following form:

$$\begin{aligned}\psi &= e^{ik_1x} \left(f_1(x, y) + \frac{g_1(x, y)}{ik_1} + \frac{h_1(x, y)}{k_1^2} + \dots \right), \\ \psi &= e^{ik_2y} \left(f_2(x, y) + \frac{g_2(x, y)}{ik_2} + \frac{h_2(x, y)}{k_2^2} + \dots \right).\end{aligned}$$

2) on $\Gamma \setminus \{P_1, P_2\}$ the function ψ is meromorphic with simple poles on γ .

3) $\psi(x, y, r) = d$, $d \in \mathbb{C}$.

For the spectral data in general position there is unique Baker–Akhiezer function.

Let us express the Baker–Akhiezer function explicitly in terms of theta function of the surface Γ .

On the surface Γ , choose a basis of cycles

$$a_1, \dots, a_g, b_1, \dots, b_g$$

with the following intersections indices:

$$a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}.$$

By $\omega_1, \dots, \omega_g$ we denote a basis of holomorphic differentials on Γ that are normalized by the conditions

$$\int_{a_j} \omega_i = \delta_{ij}.$$

Denote the matrix of b -periods of the differentials ω_j with the components

$$B_{ij} = \int_{b_i} \omega_j$$

by B . This matrix is symmetric and has a positive definite imaginary part.

The Riemann theta function is defined by the absolutely converging series

$$\theta(z) = \sum_{m \in \mathbb{Z}} e^{\pi i(Bm, m) + 2\pi i(m, z)}, \quad z = (z_1, \dots, z_g) \in \mathbb{C}^g.$$

The theta function has the following characteristics:

$$\theta(z + m) = \theta(z), \quad m \in \mathbb{Z},$$

$$\theta(z + Bm) = \exp(-\pi i(Bm, m) - 2\pi i(m, z))\theta(z), \quad m \in \mathbb{Z}.$$

Let X denote the Jacobi variety of the surface Γ :

$$X = \mathbb{C}^g / \{\mathbb{Z}^g + B\mathbb{Z}^g\}.$$

Let $A : \Gamma \rightarrow X$ be an Abel map defined by the formula

$$A(P) = \left(\int_{q_0}^P \omega_1, \dots, \int_{q_0}^P \omega_g \right), \quad P \in \Gamma,$$

$q_0 \in \Gamma$ being a fixed point.

For points $\gamma_1, \dots, \gamma_g$ in general position, according to the Riemann theorem, the function

$$\theta(z + A(P)),$$

where $z = K - A(\gamma_1) - \dots - A(\gamma_g)$, has exactly g zeros $\gamma_1, \dots, \gamma_g$ on Γ , K is the vector of Riemann constants.

Let Ω^1 and Ω^2 denote meromorphic differentials on Γ with simple poles only at the points P_1 and P_2 , respectively, and normalized by the conditions

$$\int_{a_j} \Omega^1 = \int_{a_j} \Omega^2 = 0, \quad j = 1, \dots, g.$$

Let U and V denote their vectors of b -periods:

$$U = \left(\int_{b_1} \Omega^1, \dots, \int_{b_g} \Omega^1 \right), \quad V = \left(\int_{b_1} \Omega^2, \dots, \int_{b_g} \Omega^2 \right).$$

Let $\tilde{\psi}$ denote the function

$$\tilde{\psi}(x, y, P) = \frac{\theta(A(P) + xU + yV + z)}{\theta(A(P) + z)} \exp(2\pi i x \int_{q_0}^P \Omega^1 + 2\pi i y \int_{P_1}^P \Omega^2).$$

Then the Baker–Akhiezer function has the form:

$$\psi(x, y, P) = \frac{\tilde{\psi}(x, y, P)}{\tilde{\psi}(x, y, r)} d.$$

3. MAIN THEOREM

Let $\varphi^1, \varphi^2, \varphi^3$ denote the following functions:

$$\varphi^i = \alpha_i \psi(x, y, Q_i),$$

where $Q_1, Q_2, Q_3 \in \Gamma$ is an additional set of points and α_i are some constants.

In the following paragraph (theorem 1) we find restrictions of the spectral data for the vector valued function $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ to define a Lagrangian immersion of the plane \mathbb{R}^2 in $\mathbb{C}P^2$. The spectral data in theorem 1 is a modification of the spectral data for n -orthogonal curvilinear coordinate system in \mathbb{R}^3 [10].

3.1. Lagrangian immersions. Suppose that the surface Γ has an antiholomorphic involution μ

$$\mu : \Gamma \rightarrow \Gamma,$$

for which the points P_1, P_2 and r are fixed and

$$k_i(\mu(P)) = \bar{k}_i(P).$$

The following theorem holds:

Theorem 1. *Let Q_i be fixed points of the antiholomorphic involution μ . Suppose that on Γ there exists a meromorphic 1-form Ω with the following set of divisors of zeros and poles:*

$$(\Omega)_0 = \gamma + \mu\gamma + P_1 + P_2,$$

$$(\Omega)_\infty = Q_1 + Q_2 + Q_3 + r.$$

Then the functions φ^i satisfy the equations

$$\varphi^1 \bar{\varphi}^1 A_1 + \varphi^2 \bar{\varphi}^2 A_2 + \varphi^3 \bar{\varphi}^3 A_3 + |d|^2 \text{Res}_r \Omega = 0,$$

$$\varphi^1 \bar{\varphi}_x^1 A_1 + \varphi^2 \bar{\varphi}_x^2 A_2 + \varphi^3 \bar{\varphi}_x^3 A_3 = 0,$$

$$\varphi^1 \bar{\varphi}_y^1 A_1 + \varphi^2 \bar{\varphi}_y^2 A_2 + \varphi^3 \bar{\varphi}_y^3 A_3 = 0,$$

$$\varphi_x^1 \bar{\varphi}_y^1 A_1 + \varphi_x^2 \bar{\varphi}_y^2 A_2 + \varphi_x^3 \bar{\varphi}_y^3 A_3 = 0,$$

$$\varphi_x^1 \bar{\varphi}_x^1 A_1 + \varphi_x^2 \bar{\varphi}_x^2 A_2 + \varphi_x^3 \bar{\varphi}_x^3 A_3 + |f_1|^2 c_1 = 0, \quad (4)$$

$$\varphi_y^1 \bar{\varphi}_y^1 A_1 + \varphi_y^2 \bar{\varphi}_y^2 A_2 + \varphi_y^3 \bar{\varphi}_y^3 A_3 + |f_2|^2 c_2 = 0, \quad (5)$$

where $A_k = \frac{\text{Res}_{Q_k} \Omega}{|\alpha_k|^2}$, $k = 1, 2, 3$, c_1, c_2 are the coefficients of the form Ω in the neighborhood of the points P_1 and P_2 :

$$\Omega = (c_1 w_1 + a w_1^2 + \dots) dw_1, \quad w_1 = 1/k_1,$$

$$\Omega = (c_2 w_2 + b w_2^2 + \dots) dw_2, \quad w_2 = 1/k_2.$$

The theorem 1 implies

Corollary 2. *If $\text{Res}_{Q_i} \Omega > 0$, then for*

$$\alpha_i = \sqrt{\text{Res}_{Q_i} \Omega}, \quad d = \sqrt{\frac{-1}{\text{Res}_r \Omega}},$$

the following equalities hold:

$$\langle \varphi, \varphi \rangle = 1, \quad \langle \varphi, \varphi_x \rangle = \langle \varphi, \varphi_y \rangle = \langle \varphi_x, \varphi_y \rangle = 0,$$

i.e. the mapping $\mathcal{H} \circ \varphi : \mathbb{R}^2 \rightarrow \mathbb{CP}^2$ is Lagrangian, with the induced metric on Σ having a diagonal form

$$ds^2 = |f_1|^2 |c_1| dx^2 + |f_2|^2 |c_2| dy^2.$$

Further we assume that $\text{Res}_{Q_i} \Omega > 0$ and $f_1 \neq 0, f_2 \neq 0$.

Proof of Theorem 1. Consider the 1-form $\Omega_1 = \psi(P) \overline{\psi(\mu(P))} \Omega$. By virtue of the definition of the involution μ , the function $\overline{\psi(\mu(P))}$ has the following form in the neighborhoods of the points P_1 and P_2 :

$$\begin{aligned} \overline{\psi(\mu(P))} &= e^{-ik_1 x} \left(\bar{f}_1(x, y) - \frac{\bar{g}_1(x, y)}{ik_1} + \frac{\bar{h}_1(x, y)}{k_1^2} + \dots \right), \\ \overline{\psi(\mu(P))} &= e^{-ik_2 y} \left(\bar{f}_2(x, y) - \frac{\bar{g}_2(x, y)}{ik_2} + \frac{\bar{h}_2(x, y)}{k_2^2} + \dots \right). \end{aligned}$$

Consequently, the form Ω_1 has no essential singularities at the points P_1 and P_2 . The simple poles $\gamma + \mu\gamma$ of the function $\psi(P)\overline{\psi(\mu P)}$ cancel out the zeros of the form Ω at these points. Thus, the form Ω_1 has only simple poles at the points Q_1, Q_2, Q_3 and r with the residues equal to

$$\psi(Q_1) \overline{\psi(Q_1)} \text{Res}_{Q_1} \Omega = \varphi^1 \bar{\varphi}^1 A_1, \quad \varphi^2 \bar{\varphi}^2 A_2, \quad \varphi^3 \bar{\varphi}^3 A_3, \quad |d|^2 \text{Res}_r \Omega.$$

Consequently, the sum of these residues is equal to zero, and this proves the first equality of the theorem 1.

The form $\psi(P) \overline{\psi(\mu(P))}_x \Omega$ has no essential singularities at the points P_1 and P_2 either. This form has only simple poles at the points Q_1, Q_2 and Q_3 with the residues equal to

$$\varphi^1 \bar{\varphi}_x^1 A_1, \quad \varphi^2 \bar{\varphi}_x^2 A_2, \quad \varphi^3 \bar{\varphi}_x^3 A_3.$$

The second equality is proven. The proof of the next two equalities is analogous. It is based on the analysis of the forms

$$\psi(P) \overline{\psi(\mu(P))}_y \Omega, \quad \psi(P)_x \overline{\psi(\mu(P))}_y \Omega,$$

which also have only simple poles at the points Q_1, Q_2, Q_3 .

In order to prove the last two equalities (4) and (5), consider the forms

$$\psi(P)_x \overline{\psi(\mu(P))}_x \Omega, \quad \psi(P)_y \overline{\psi(\mu(P))}_y \Omega.$$

These forms have simple poles at the points Q_1, Q_2, Q_3, P_1 and Q_1, Q_2, Q_3, P_2 with the residues

$$\varphi_x^1 \bar{\varphi}_x^1 A_1, \quad \varphi_x^2 \bar{\varphi}_x^2 A_2, \quad \varphi_x^3 \bar{\varphi}_x^3 A_3, \quad |f_1|^2 c_1$$

and

$$\varphi_y^1 \bar{\varphi}_y^1 A_1, \quad \varphi_y^2 \bar{\varphi}_y^2 A_2, \quad \varphi_y^3 \bar{\varphi}_y^3 A_3, \quad |f_2|^2 c_2.$$

Theorem 1 is proven.

3.2. Minimal Lagrangian immersions. In this subsection we find spectral data such that the mapping φ , constructed in the previous subsection is minimal.

Suppose that the curve Γ has the holomorphic involution

$$\sigma : \Gamma \rightarrow \Gamma.$$

Let τ denote the composition $\mu \circ \sigma$. The following lemma holds.

Lemma 2. *Suppose that the reality conditions fulfilled*

$$\mu(\gamma) = \gamma, \quad \mu(r) = r, \quad d \in \mathbb{R}.$$

Then

$$\psi(x, y, \tau(P)) = \overline{\psi(x, y, P)}.$$

To prove this standard lemma, it is sufficient to note, that the function $\overline{\psi(x, y, \tau(P))}$ satisfies conditions **1)–3)** in the definition of the Baker–Akhiezer function as the function $\psi(x, y, P)$, consequently, the functions $\overline{\psi(x, y, \tau(P))}$ and $\psi(x, y, P)$ coincide.

Below we assume that the conditions of Lemma 2 are fulfilled. In particular, this means that

$$\mu(\gamma) = \sigma(\gamma), \quad \mu(r) = \sigma(r),$$

and that the functions f_i, g_i , which participate in the decomposition of ψ in the neighborhoods of the points P_1 and P_2 are real.

Consider three functions

$$\begin{aligned} F_{11}(x, y, P) &= \partial_x^2 \psi + \Gamma_{11}^1(x, y) \partial_x \psi + \Gamma_{11}^2(x, y) \partial_y \psi + b_{11}(x, y) \psi, \\ F_{12}(x, y, P) &= \partial_x \partial_y \psi + \Gamma_{12}^1(x, y) \partial_x \psi + \Gamma_{12}^2(x, y) \partial_y \psi + b_{12}(x, y) \psi, \\ F_{22}(x, y, P) &= \partial_y^2 \psi + \Gamma_{22}^1(x, y) \partial_x \psi + \Gamma_{22}^2(x, y) \partial_y \psi + b_{22}(x, y) \psi. \end{aligned}$$

Choose functions $\Gamma_{ij}^k(x, y)$ and $b_{ij}(x, y)$ such that

$$F_{11}(x, y, Q_i) = F_{12}(x, y, Q_i) = F_{22}(x, y, Q_i) = 0, \quad i = 1, 2, 3.$$

The following lemma holds

Lemma 3. *The following equalities hold:*

$$\begin{aligned} \Gamma_{11}^1(x, y) &= -\frac{ia}{c_1} - \frac{f_{1x}}{f_1}, \\ \Gamma_{12}^1(x, y) &= -\frac{f_{1y}}{f_1}, \quad \Gamma_{12}^2(x, y) = -\frac{f_{2x}}{f_2}, \\ \Gamma_{22}^2(x, y) &= -\frac{ib}{c_2} - \frac{f_{2y}}{f_2}. \end{aligned}$$

Proof of lemma 3. Consider the form

$$\omega = F_{11}(P) \psi(\sigma(P))_x \Omega.$$

The form ω has no essential singularities at the points P_1 and P_2 . The form ω has only a pole of the second order at P_1

$$\text{Res}_{P_1} \omega = f_1((ia + c_1 \Gamma_{11}^1) f_1 + c_1 f_{1x}) = 0.$$

This yields the formula for Γ_{11}^1 . Similarly to find other coefficients it is necessary to consider the forms

$$F_{12}(P) \psi(\sigma(P))_x \Omega, \quad F_{12}(P) \psi(\sigma(P))_y \Omega, \quad F_{22}(P) \psi(\sigma(P))_y \Omega.$$

Lemma 3 is proven.

This lemma implies

$$\Gamma_{11}^1 + \Gamma_{12}^2 = -\frac{ia}{c_1} - \frac{f_{1x}}{f_1} - \frac{f_{2x}}{f_2},$$

$$\Gamma_{12}^1 + \Gamma_{22}^2 = -\frac{ib}{c_2} - \frac{f_{1_y}}{f_1} - \frac{f_{2_y}}{f_2}.$$

Suppose that $a = b = 0$. Then as f_1 and f_2 are real functions, so by Corollary 1 the surface is minimal. Thus the main theorem holds

Theorem 2. *Suppose that the spectral curve Γ has the antiholomorphic involution*

$$\mu : \Gamma \rightarrow \Gamma$$

with fixed points Q_1, Q_2, Q_3, P_1, P_2 and meromorphic 1-form Ω with the following divisors of zeros and poles

$$(\Omega)_0 = \gamma + \mu\gamma + P_1 + P_2, \quad (\Omega)_\infty = Q_1 + Q_2 + Q_3 + r,$$

and $\text{Res}_{Q_i}\Omega > 0$. Then the mapping $\mathcal{H} \circ \varphi$, where $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ gives Lagrangian mapping of the plane in \mathbb{CP}^2 .

Besides let the spectral curve Γ has the holomorphic involution

$$\sigma : \Gamma \rightarrow \Gamma$$

such that

$$\mu(\gamma) = \sigma(\gamma), \quad \tau(r) = r$$

and $d \in \mathbb{R}$ and suppose, that the form Ω has the following decomposition in the neighborhood of the points P_1, P_2

$$\Omega = (c_1 w_1 + d_1 w_1^3 + \dots) dw_1, \quad w_1 = 1/k_1, \quad (6)$$

$$\Omega = (c_2 w_2 + d_2 w_2^3 + \dots) dw_2, \quad w_2 = 1/k_2, \quad (7)$$

then the mapping is minimal.

3.3. Examples. In this paragraph we demonstrate the example of theorem 2, when the spectral curve Γ is reducible and consists of irreducible components Γ_i , which are isomorphic \mathbb{CP}^1 . In this case the theorem 2 is also valid, but in definition of Baker–Akhiezer function the genus need to be changed on arithmetical genus, and in the formulation of theorem 2 the differential need to be changed on the differential, which satisfies the condition of regularity at the points of intersection of different components.

The regular differential on Γ is defined by meromorphic 1-forms Ω_j on Γ_j with simple poles. The poles of the forms Ω_j are allowed just in the points of components' intersections. And the conditions of regularity must be fulfilled: if the curves Γ_i and Γ_j intersect at the point P , then

$$\text{Res}_P \Omega_1 + \text{Res}_P \Omega_2 = 0.$$

The arithmetical genus of the curve Γ is called the dimension of the space of the regular differentials. A number of poles γ_i in the definition of Baker–Akhiezer function must coincide with the arithmetical genus of the curve Γ (see [11]).

Let the curve Γ consists of two components Γ_1 and Γ_2 , which intersect at two points.

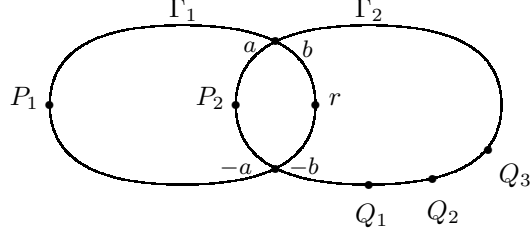


Figure 1.

Let z_1 be a coordinate on the first component, z_2 be a coordinate on the second component. Suppose that, the points of the intersection on the first component have coordinates $a, -a \in \mathbb{R}$, and on the second $b, -b \in \mathbb{R}$. Let

$$P_1 = \infty \in \Gamma_1, \quad P_2 = \infty \in \Gamma_2, \quad r = 0 \in \Gamma_1,$$

$$Q_1, Q_2, Q_3 \in \Gamma_2, \quad Q_i \in \mathbb{R}, \quad \gamma \in \Gamma_2, \gamma = i\Gamma \in i\mathbb{R}.$$

The curve Γ has the holomorphic involution

$$\sigma : \Gamma \rightarrow \Gamma, \quad \sigma(z_1) = -z_1, \quad \sigma(z_2) = -z_2.$$

and antiholomorphic involution

$$\mu : \Gamma \rightarrow \Gamma, \quad \mu(z_1) = \bar{z}_1, \quad \mu(z_2) = \bar{z}_2.$$

Baker–Akhiezer function ψ on Γ is defined by the functions ψ_1 and ψ_2 on the components Γ_1 and Γ_2

$$\psi_1 = e^{ixz_1} f_1(x, y), \quad \psi_2 = e^{iyz_2} \left(f_2(x, y) + \frac{g_2(x, y)}{z_2 - \gamma} \right).$$

The functions f_1, f_2 and g are found from the consistency conditions

$$\psi_1(x, y, a) = \psi_2(x, y, b), \quad \psi_1(x, y, -a) = \psi_2(x, y, -b),$$

and normalization condition

$$\psi_1(x, y, 0) = d.$$

Hence

$$f_1 = d, \quad f_2 = \frac{de^{-i(ax+by)}}{2b} (b(e^{2iax} + e^{2iby}) + \gamma(-e^{2iax} + e^{2iby})),$$

$$g_2 = \frac{de^{-i(ax+by)}}{2b} (e^{2iax} + e^{2iby})(b^2 - \gamma^2).$$

The meromorphic form Ω is defined by the forms

$$\Omega_1 = \frac{dz_1}{z_1(z_1^2 - a^2)}, \quad \Omega_2 = \frac{c_1(z_2^2 - \gamma^2)dz_2}{(z_2 - Q_1)(z_2 - Q_2)(z - Q_3)(z_2^2 - b^2)},$$

therefore

$$d = \sqrt{\frac{1}{|\text{Res}_0 \Omega_1|}} = a.$$

Since

$$\text{Res}_a \Omega_1 + \text{Res}_b \Omega_2 = 0, \quad \text{Res}_{-a} \Omega_1 + \text{Res}_{-b} \Omega_2 = 0$$

we get

$$c_1 = -\frac{b(b-Q_1)(b-Q_2)(b-Q_3)}{a^2(b^2-\gamma^2)},$$

$$Q_3 = -\frac{b^2(Q_1+Q_2)}{b^2+Q_1Q_2}.$$

From the condition that the form Ω has a expansions (6) and (7) in the neighborhoods P_1 and P_2 we get $Q_2 = Q_1$. Hence the components of the mapping φ have the form

$$\begin{aligned}\varphi_1 &= \alpha_1 F_2(Q_1) = \\ &= \frac{\alpha_1 a e^{-i(ax+(b-Q_1)y)} (e^{2iax}(b+Q_1)(b-i\Gamma) - e^{2iby}(b-Q_1)(b+i\Gamma))}{2b(Q_1-i\Gamma)}, \\ \varphi_2 &= \alpha_2 F_2(Q_2) = \\ &= \frac{\alpha_2 a e^{-i(ax+(b+Q_1)y)} (e^{2iax}(b-Q_1)(b-i\Gamma) - e^{2iby}(b+Q_1)(b+i\Gamma))}{2b(-Q_1-i\Gamma)}, \\ \varphi_3 &= \alpha_2 F_2(Q_3) = \alpha_3 a e^{-i(ax+\frac{b(b+Q_1)(b+Q_2)}{b^2+Q_1Q_2}y)} \times \\ &= \frac{(-be^{2iax}(b-Q_1)(b-Q_2)(b-i\Gamma) + be^{2iby}(b+Q_1)(b+Q_2)(b+i\Gamma))}{2(b^3(Q_1+Q_2+i\Gamma) + ibQ_1Q_2\Gamma)},\end{aligned}$$

where

$$\begin{aligned}\alpha_1 &= \sqrt{\text{Res}_{Q_1}\Omega_2} = \sqrt{\frac{b^2(Q_1^2+\Gamma^2)}{2a^2Q_1^2(b^2+\Gamma^2)}}, \\ \alpha_2 &= \sqrt{\text{Res}_{Q_2}\Omega_2} = \sqrt{\frac{b^2(Q_1^2+\Gamma^2)}{2a^2Q_1^2(b^2+\Gamma^2)}}, \\ \alpha_3 &= \sqrt{\text{Res}_{Q_3}\Omega_2} = \sqrt{\frac{\Gamma^2(Q_1^2-b^2)}{a^2Q_1^2(b^2+\Gamma^2)}}.\end{aligned}$$

For $a = b = 1, Q_1 = 2, \gamma = i$ we get

$$\begin{aligned}\varphi_1 &= \frac{(1+3i)}{8\sqrt{5}} e^{-i(x-y)} (-3ie^{2ix} + e^{2iy}), \\ \varphi_2 &= \frac{e^{-i(x+3y)}}{8\sqrt{5}} ((1-3i)e^{2ix} + (9+3i)e^{2iy}), \\ \varphi_3 &= \frac{1}{2} \sqrt{\frac{1}{2}} (\cos(x-y) - \sin(x-y)),\end{aligned}$$

and meanwhile

$$e^{2i\beta} = -1.$$

Induced metric on the image has the form

$$ds^2 = dx^2 + \frac{3}{2}(1 + \sin(2(x-y)))dy^2.$$

The Gaussian curvature of the surface is equal to 1. Hence the image of the mapping is sphere.

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